Bohmian transmission and reflection dwell times without trajectory sampling

Bohmian transmission and reflection dwell times without trajectory sampling

Sabine Kreidl

Institute for Theoretical Physics, University of Innsbruck, Technikerstraße 25, A-6020 Innsbruck, Austria

E-mail: sabine.kreidl@uibk.ac.at

Abstract
Within the framework of Bohmian mechanics dwell times find a straightforward formulation. The computation of associated probabilities and distributions however needs the explicit knowledge of a relevant sample of trajectories and therefore implies formidable numerical effort. Here, a trajectory free formulation for the average transmission and reflection dwell times within static spatial intervals $[a, b]$ is given for one-dimensional scattering problems. This formulation reduces the computation time to less than 5% of the computation time by means of trajectory sampling.

PACS number: 03.65.−w

(Some figures in this article are in colour only in the electronic version)
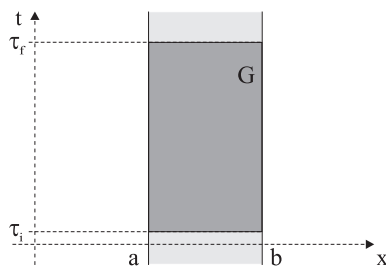
1. Introduction

For a 1D static detector located in the spatial interval $[a, b]$ (see figure 1), the *average dwell time* of an ensemble of quantum systems with wavefunction Ψ from time τ_i up to time τ_f is generally agreed to be given by

$$\int_{\tau_i}^{\tau_f} \mathrm{d}t \int_a^b \mathrm{d}\xi \, |\Psi(t, \xi)|^2, \tag{1}$$

cf the reviews [1][1]. It is motivated by classical reasoning [2], and it also has been derived within Bohmian mechanics [3]. In a recent work [4], a corresponding dwell time operator has been investigated.

Dwell times of the type (1) were associated with interaction times in collision processes long ago [5]. The relevance of these interaction times to time-resolved scattering experiments has been studied, e.g., in [6]. Through these works it became clear that *differences* of average

---

[1] Clearly expression (1) does not exist for bound states.

Figure 1. Spacetime region $G = [a, b] \times [\tau_i, \tau_f]$.

dwell times formed between a scattering wave packet and its free incoming asymptote are measurable quantities.

With the introduction of the Larmor clock by Baz [7], dwell time expressions themselves have got measurable status. The idea is that a small and uniform magnetic field, which is confined to a small region of space, causes a Larmor precession of the spin-polarization vector of the scattered wave. It was shown in [8] that the Larmor clock indeed reveals the average dwell time. If it also is capable of displaying the spectral distribution of some dwell time operator is still unclear.

The specialization of the Larmor clock to the case of one-dimensional scattering was done by Rybachenko in [9]. In this work (to the authors knowledge for the first time), a distinction into the dwell times of the finally transmitted respectively reflected partial waves has been introduced. For short such selective dwell times will further on be denoted as transmission and reflection times. Another approach to transmission and reflection times, grounded on a specific experimental scheme, was introduced by the oscillating barrier model of Büttiker and Landauer [10]. This model is widely believed to have ignited the tunnelling time controversy anew. A further extension of the Larmor clock by Büttiker [11] incorporates the effect of spin alignment with the magnetic field.

More recently the interest in transmission times has been driven on the one hand by the very indirect time measurement techniques of the condensed matter community, especially in connection with tunnelling in semiconductor heterostructures or Josephson junctions. On the other hand, the progress in laser cooling techniques in quantum optics delivers another valuable tool for future time-resolved scattering experiments.

It is no surprise that the detailed definition of transmission and reflection times depends on the situation under scrutiny. A systematic operator approach embodying several such possible definitions was given by Brouard *et al* [12]. It was shown, however, that transmission and reflection times derived within the framework of Bohmian mechanics are not included in this catalogue ([13], chapter 5).

Bohmian mechanics comprises the mathematical concept of worldlines or trajectories and with this the term 'particle' obtains substance in quantum theory again. Thus the notion of dwell time can be addressed in a straightforward manner, very much like in classical mechanics. The dwell time of a particle in the spatial interval $[a, b]$ is simply defined as the duration during which the particle's trajectory is localized within $[a, b]$. For an elaborate discussion of Bohmian mechanics see [14].

As the numerical effort involved with the calculation of Bohmian worldlines is immense, there have been attempts to compute the Bohmian transmission and reflection times without the sampling of trajectories. A related one-dimensional bound-state situation has, e.g., been studied by Stomphorst in [15]. In the present paper, a formulation without trajectories for the

transmission and reflection times in a genuine scattering situation is presented. The derivation closely follows ideas developed for the treatment of 1D arrival time in [16]. As an application the scattering from a double potential barrier, reminiscent of semiconductor heterojunctions in, e.g., resonant tunnelling diodes, is considered.

2. Bohmian transmission and reflection times in 1D scattering situations

We consider a scattering situation in which the Møller operators Ω^{in} and Ω^{out} exist and are asymptotically complete [17]. $\Psi(t, \cdot) = e^{-iHt/\hbar}\Omega^{\text{in}}\phi_0 = \Omega^{\text{in}} e^{-iH_0t/\hbar}\phi_0$ denotes the scattering solution with incoming asymptote $\phi(t, \cdot) = e^{-iH_0t/\hbar}\phi_0$. The solution $\phi(t, \cdot)$ of the free Schrödinger equation is chosen to be localized on the negative spatial semi-axis for $t \to -\infty$. That is the case if and only if the Fourier transform $\mathcal{F}(\phi_0)$ is localized on the positive half-line [18].

The Bohmian transmission time in the above scattering situation is defined as follows. Let $\gamma_x(t)$ denote the Bohmian trajectory which at time $t = 0$ passes through the point x. There exists a critical value $x_c \in \mathbb{R}$, such that $\lim_{t\to\infty} \gamma_x(t) = -\infty$ for all $x < x_c$ and that $\lim_{t\to\infty} \gamma_x(t) = \infty$ for all $x > x_c$. Therefore, the Bohmian transmission time is defined as

$$\langle \tau_T \rangle = \int_{-\infty}^{\infty} dx\, |\Psi(0, x)|^2 \cdot \Theta(x - x_c) \int_{\tau_i}^{\tau_f} dt\, \chi_{[a,b]}(\gamma_x(t)) \tag{2}$$

with $\chi_{[a,b]}$ the characteristic function on the interval $[a, b]$ and Θ the Heaviside step function. See, e.g., [19]. Analogously the reflection time $\langle \tau_R \rangle$ is defined by replacing the term $\Theta(x - x_c)$ in the right-hand side of equation (2) by $\Theta(x_c - x)$. The critical trajectory $\gamma_{x_c}(t)$ $\left(\gamma_{x_c}(0) = x_c\right)$ is implicitly defined by

$$|T|^2 = \int_{\gamma_{x_c}(t)}^{\infty} |\Psi(t, \xi)|^2\, d\xi, \qquad \forall t \in \mathbb{R}. \tag{3}$$

Thereby,

$$|T|^2 := \lim_{t\to\infty} \int_0^{\infty} |\Psi(t, \xi)|^2\, d\xi \tag{4}$$

is the transmission probability of the scattering system. The lower limit of the integral in (4) can equally be replaced by any finite $q \in \mathbb{R}$. Accordingly by $|R|^2 := 1 - |T|^2$, the reflection probability is defined. Thus the conditional transmission respectively reflection times, i.e. transmission and reflection times normalized to the fraction of transmitted respectively reflected particles of the entire ensemble, are $\langle \tau_T \rangle^c := \frac{1}{|T|^2}\langle \tau_T \rangle$ and $\langle \tau_R \rangle^c := \frac{1}{|R|^2}\langle \tau_R \rangle$.

From the computational viewpoint, the two terms $\langle \tau_T \rangle$ and $\langle \tau_R \rangle$ by means of Bohmian mechanics are achieved in a straightforward manner. One chooses an appropriate sample of initial values on the configuration space $\Sigma_0 = \mathbb{R}$, calculates the corresponding trajectories over a sufficient range of time, determines the dwell time for each trajectory, labels the trajectories as transmitted or reflected according to their position at large times and finally calculates, according to the weight of each trajectory and just as in classical statistics, the average times. That this programme involves formidable numerical effort is evident.

However, the calculation of Bohmian transmission and reflection times can be reduced to the computation of current density integrals along the edges at $x = a$ and $x = b$. The next proposition represents a generalization of expressions already proposed in [20].

Proposition 1. *For 1D scattering solutions $\Omega^{\text{in}}\phi_t$ with $\Theta(K)\phi_0 = \phi_0$, $\|\phi_0\| = 1$, for the Bohmian transmission and reflection times within $[a, b] \times [\tau_i, \tau_f]$, hold*

$$\langle \tau_T \rangle = \int_{\tau_i}^{\tau_f} dt\, [\min\{f_a(t), |T|^2\} - \min\{f_b(t), |T|^2\}] \tag{5}$$

and

$$\langle \tau_R \rangle = \int_{\tau_i}^{\tau_f} dt \, [\max\{f_a(t), |T|^2\} - \max\{f_b(t), |T|^2\}] \qquad (6)$$

with

$$f_q(t) := \int_{-\infty}^{t} j(s, q) \, ds.$$

Here j is the quantum mechanical probability current density.

An essential ingredient for the proof of proposition 1 is the relation

$$f_q(t) := \int_{-\infty}^{t} j(s, q) \, ds = \int_{q}^{\infty} |\Psi(t, \xi)|^2 \, d\xi. \qquad (7)$$

It depicts that the probability of finding a particle to the right of q at time t is equal to the amount of probability, which has passed q up to the time t. A plausibility argument for equation (7) and a rigorous proof for a limited class of scattering situations is given in appendix A. The proof of proposition 1 is given in appendix B.

Remark. The formulation of Oriols *et al* [20] assumes the case in which the current density at the right edge of the barrier does not change its sign and is positive for all times. In this case $f_b(t) \leqslant |T|^2, \forall t \in \mathbb{R}$, because

$$|T|^2 \overset{(4)}{=} \lim_{t \to \infty} \int_{b}^{\infty} |\Psi(t, \xi)|^2 \, d\xi \overset{(7)}{=} \lim_{t \to \infty} \int_{-\infty}^{t} j(s, b) \, ds = \int_{-\infty}^{\infty} j(s, b) \, ds$$

and further

$$f_b(t) = \int_{-\infty}^{t} j(s, b) \, ds \overset{j(\cdot, b) \geqslant 0}{\leqslant} \int_{-\infty}^{\infty} j(s, b) \, ds = |T|^2.$$

Then equation (5) reduces to

$$\langle \tau_T \rangle^c = \frac{1}{|T|^2} \langle \tau_T \rangle = \frac{1}{|T|^2} \int_{\tau_i}^{\tau_f} dt \, [\min\{f_a(t), |T|^2\} - f_b(t)]$$

and equation (6) to

$$\langle \tau_R \rangle^c = \frac{1}{|R|^2} \langle \tau_R \rangle = \frac{1}{|R|^2} \int_{\tau_i}^{\tau_f} dt \, [\max\{f_a(t), |T|^2\} - |T|^2]$$

which reproduces equations (14) and (15) of [20] for the special choices $\tau_i = 0$ and $\tau_f = \infty$.

Obviously an interesting task would be to construct a device, i.e. a clock, which measures the Bohmian transmission and reflection times. Such a clock should display the respective time, let us say, through the Bohmian centre of mass position of its hand at the instant of its readout, which presumably has to be chosen by the experimenter. The combined system's wavefunction would be modelled by an appropriate Schrödinger equation, incorporating the interaction between the micro-system and the clock. There is a self-adjoint operator corresponding to the hand's positions. The crucial question is whether this observable's spectral distribution in a given state at the instant of its readout coincides with the respective Bohmian transmission (reflection) time distribution. Probably this is not the case for all states. However, similarly to the issue of exit time statistics [21], a subspace might be identified on which the Bohmian distribution coincides with one of the standard quantum mechanical distributions.
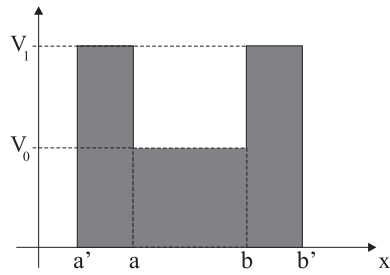
Figure 2. Double potential barrier.
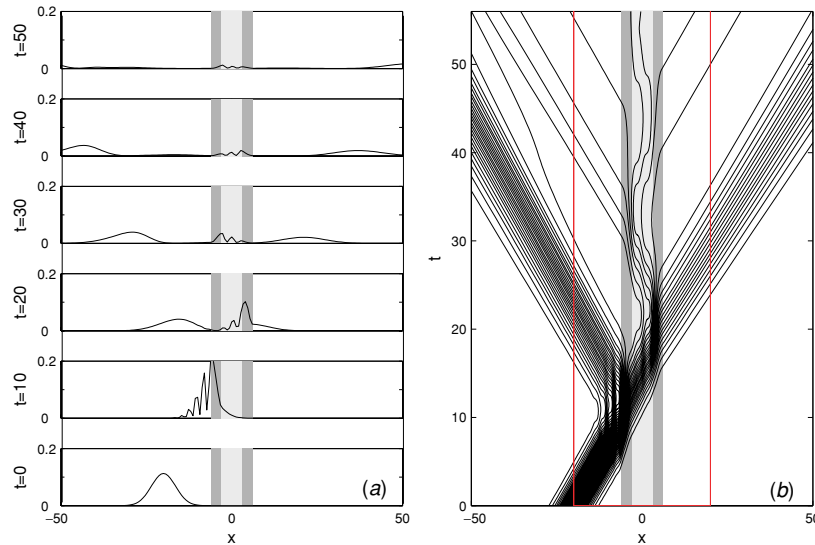
Figure 3. (*a*) Evolution of the probability density $|\Psi|^2$. (*b*) Corresponding sample of 50 Bohmian trajectories. The double potential barrier is indicated by the hatched area.

3. Numerical example: transmission and reflection at a double potential barrier structure

As an example for a 1D scattering situation we consider the case of a Gaussian wave packet impinging on a double potential barrier, i.e. we are looking for solutions Ψ to the Schrödinger equation

$$i\hbar\partial_t\Psi = \left[-\frac{\hbar^2}{2m}\partial_x^2 + V\right]\Psi$$

with

$$V = V_0\chi_{[a,b]} + V_1(\chi_{[a',a[}{}^+\chi_{]b,b']}).$$

$\chi_{[\alpha,\beta]}$ denotes the characteristic function on the interval $[\alpha,\beta] \in \mathbb{R}$ and $a' < a < b < b'$ (see figure 2). The parameter reduction $\hbar, m, V_0 \rightarrow 1$, e.g., is achieved by taking time in units of $\frac{\hbar}{2V_0}$, space in units of $\frac{\hbar}{\sqrt{2mV_0}}$. V_1 is taken in units of V_0. In figure 3(*a*) the evolution of the probability density $|\Psi|^2$ is given for the case $a' = -6, a = -3, b = 3, b' = 6$ and $V_1 = 2$ in the chosen units. The mean kinetic energy of the packet is $1.5^2V_0 = 2.25V_0$. In figure 3(*b*), a sample of 50 corresponding Bohmian trajectories is illustrated. The initial distribution of
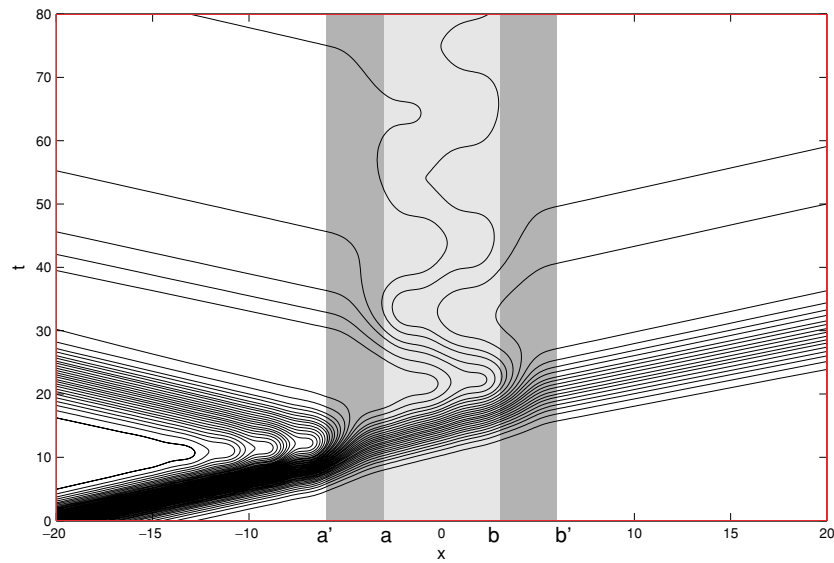
**Figure 4.** Zoom into the region indicated in figure 3(b).
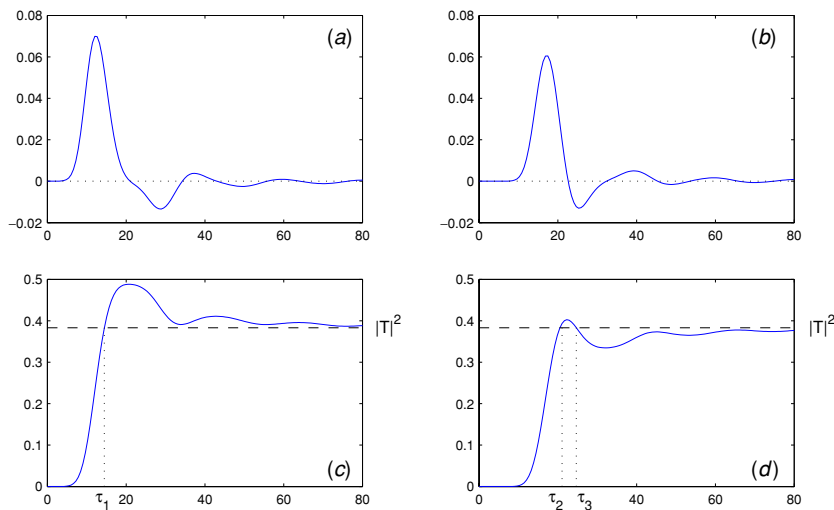
Figure 5. Current densities $j(\cdot, a)$ (a) and $j(\cdot, b)$ (b) and corresponding integrated current densities $f_a(\cdot)$ (c) and $f_b(\cdot)$ (d).

starting points of the trajectories resemble the initial Gaussian distribution of the wave packet. Figure 4 shows a zoom into the area indicated by the rectangle in figure 3(b).

In figure 4 it becomes clear that, as trajectories change their direction at $x = b$, the current density in this case changes its sign also at the right edge of the area in question. Therefore the restricted formulae of Oriols *et al* lose their validity and the generalized expressions (5) and (6) have to be applied.

Figure 5 shows the current densities and corresponding integrated current densities at the edges $x = a$ and $x = b$, respectively, as a function of time. In (c) and (d), the transmission coefficient $|T|^2 = 38.31\%$ is indicated by the dashed line.
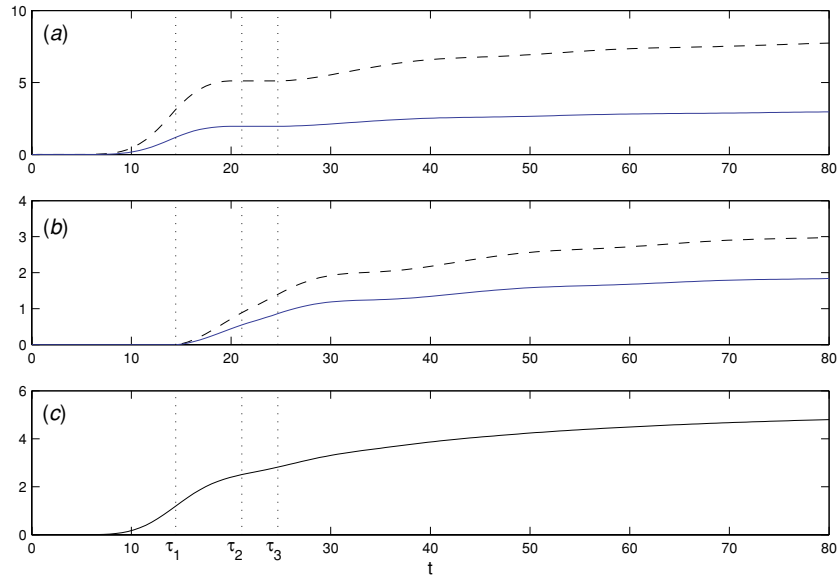
Figure 6. (*a*) Transmission times $\mathcal{T}_T(t)$ (solid line) and corresponding conditional transmission times (dashed line). (*b*) Reflection times $\mathcal{T}_R(t)$ (solid line) and conditional reflection times (dashed line). (*c*) Average dwell times $\mathcal{T}_D(t)$ inside $[a, b] \times [0, t]$.

Finally, in figure 6 the mappings $\mathcal{T}_T(s)$ (*a*), $\mathcal{T}_R(s)$ (*b*) and $\mathcal{T}_D(s)$ (*c*) are illustrated, which give the Bohmian transmission, reflection and overall average dwell times $\langle \tau_X \rangle$, $X \in \{T, R, D\}$ in $[a, b]$ from time $\tau_i = 0$ onwards as a function of the upper temporal bound $s = \tau_f$. In addition the conditional transmission and reflection times, i.e. normalized to the fraction of finally transmitted or reflected particles, are indicated in (*a*) and (*b*).

Introducing as parameters the potential energy $V_0 = 0.25$ eV (i.e. $V_1 = 0.5$ eV) and the effective electron mass $m_* = 0.07 m_e$ typical for GaAs/GaAlAs double barrier heterostructures (see, e.g., [22]), the temporal units are approximately 1.3 fs, the spatial units approximately 15 Å. Therefore, the width of the model heterostructure in figure 2 is in the range of 200 Å, which is easily achieved by the ultrathin layers of modern semiconductor devices.

With the aid of formulae (5) and (6) of proposition 1 the computational effort involved with the calculation of transmission and reflection times was reduced to about 5% of that corresponding to the computation by means of trajectory sampling.

Acknowledgments

Appendix A. Proof of equality (7)

Recall that the scattering solutions under consideration are given by $\Psi(t, \cdot) = \Omega^{\text{in}} \phi(t, \cdot)$, with $\phi(t, \cdot) = \text{e}^{-\text{i}H_0 t/\hbar} \phi_0$ a solution to the free Schrödinger equation. The Fourier transform $\varphi := \mathcal{F}(\phi_0)$ is assumed to be localized exclusively on the positive half line, for which reason
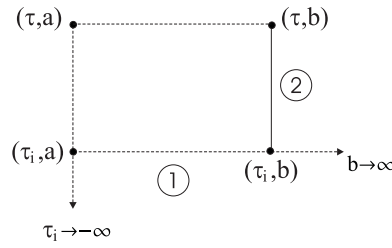
Figure 7. Closed spacetime region.

$\phi(t, \cdot)$ will be further and further localized to the left for large negative times (in the sense of the L^2-norm $\|\cdot\|$).

Figure 7 then illustrates, that for the scattering wave packet Ψ, formula (7) is a plausible conjecture. In the following a rigorous proof will be given for a limited class of scattering solutions.

For the closed spacetime region in figure 7 integration of the continuity equation assures that

$$\int_a^b |\Psi(\tau, \xi)|^2 \, \mathrm{d}\xi = \int_{\tau_i}^{\tau} j(s, a) \, \mathrm{d}s + \underbrace{\int_a^b |\Psi(\tau_i, \xi)|^2 \, \mathrm{d}\xi}_{①} - \underbrace{\int_{\tau_i}^{\tau} j(s, b) \, \mathrm{d}s}_{②}.$$

The aim is to show that the terms ① and ② converge towards zero in the limits $\tau_i \to -\infty$ and $b \to \infty$, in which case formula (7) applies.

Proposition 2. *For every scattering solution Ψ with the above properties, term ① vanishes in the limits $\tau_i \to -\infty$ and $b \to \infty$, i.e.,*

$$\lim_{\tau_i \to -\infty} \lim_{b \to \infty} ① = \lim_{M \to \infty} \int_a^M |\Psi(-M, \xi)|^2 \, \mathrm{d}\xi = 0. \tag{A.1}$$

Proof. First note that the inequalities

$$0 \leqslant \int_a^M |\Psi(-M, \xi)|^2 \, \mathrm{d}\xi \leqslant \int_a^{\infty} |\Psi(-M, \xi)|^2 \, \mathrm{d}\xi \tag{A.2}$$

hold. For the given scattering solution with $\lim_{t \to -\infty} \|\Psi(t, \cdot) - \phi(t, \cdot)\| = 0$, according to Dollard [18], the relation

$$\lim_{t \to -\infty} \int_A |\Psi(t, \xi)|^2 \, \mathrm{d}\xi = \lim_{t \to -\infty} \int_A |C_t(\phi_0)(\xi)|^2 \, \mathrm{d}\xi$$

holds for every measurable set $A \subseteq \mathbb{R}$ and with

$$C_t(\phi_0)(x) := \left(\frac{m}{\mathrm{i}\hbar t}\right)^{1/2} \mathrm{e}^{\mathrm{i}mx^2/2\hbar t} \varphi\left(\frac{mx}{\hbar t}\right).$$

With this, inequality (A.2) and supp$\{\varphi\} \subseteq \mathbb{R}_+$ one immediately proves (A.1) by

$$\lim_{\tau_i \to -\infty} \int_a^{\infty} |\Psi(\tau_i, \xi)|^2 \, \mathrm{d}\xi = \lim_{\tau_i \to -\infty} \int_a^{\infty} |C_{\tau_i}(\phi_0)(\xi)|^2 \, \mathrm{d}\xi$$

$$= \lim_{\tau_i \to -\infty} \left|\frac{m}{\hbar \tau_i}\right| \left|\int_a^{\infty} \left|\varphi\left(\frac{m\xi}{\hbar \tau_i}\right)\right|^2 \, \mathrm{d}\xi = \lim_{\tau_i \to -\infty} \int_{-\infty}^{\frac{ma}{\hbar \tau_i}} |\varphi(k)|^2 \, \mathrm{d}k$$

$$= \int_{-\infty}^0 |\varphi(k)|^2 \, \mathrm{d}k = 0. \qquad \square$$

Proposition 3. *For a freely evolving wave packet $\phi(t, \cdot) = \mathrm{e}^{-\mathrm{i}H_0 t/\hbar}\phi_0$ with Fourier transform $\varphi := \mathcal{F}(\phi_0), \varphi \in C^1(\mathbb{R})$ and $\mathrm{supp}\{\varphi\} \subseteq [a_1, a_2], 0 < a_1 < a_2$ (i.e. $\varphi \in C_0^1(\mathbb{R}_+)$) term ② vanishes in the limits $\tau_i \to -\infty$ and $b \to \infty$, i.e.,*

$$\lim_{\tau_i \to -\infty} \lim_{b \to \infty} ② = \lim_{M \to \infty} \int_{-M}^{\tau} j(s, M)\,\mathrm{d}s = 0. \tag{A.3}$$

Proof.   Equation ($A.3$) can immediately be shown by a stationary phase argument. In the following, the parameter reduced notation $\hbar = m = 1$ will be used.

The solution ϕ of the free Schrödinger equation can be written in the form

$$\phi(t, x) = \mathcal{F}^{-1}(\mathrm{e}^{-\mathrm{i}\omega t}\mathcal{F}(\phi_0))(x) = (2\pi)^{-1/2} \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\varphi(k)$$

with the function $\omega : \mathbb{R} \to \mathbb{R}, k \mapsto k^2/2$ being in $C^\infty(\mathbb{R})$. The stationary phase argument then states (cf, e.g., [17], appendix 1 to XI.3) that for every open set $A \supseteq [a_1, a_2] \supseteq \{\omega'(k)|k \in \mathrm{supp}\{\varphi\}\}$, there is a constant $C > 0$ such that for all $(t, x) \in \mathbb{R}^2$ with $\frac{x}{t} \notin A$

$$|\phi(t, x)| \leqslant C(1 + |x| + |t|)^{-1}.$$

The same considerations then deliver a second constant $C' > 0$ such that for all $(t, x) \in \mathbb{R}^2$ with $\frac{x}{t} \notin A$

$$\left|\frac{\partial}{\partial x}\phi(t, x)\right| = \left|\int \mathrm{d}k(\mathrm{i}k)\, \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\varphi(k)\right| \leqslant C'(1 + |x| + |t|)^{-1}.$$

We choose without loss of generality $A := \left]\frac{a_1}{2}, 2 \cdot a_2\right[$. Then for a fixed $\tau \in \mathbb{R}$ there is an $M > 0$ such that for all $t \leqslant \tau$ and all $x \geqslant M$: $\frac{x}{t} \notin A$ (set, e.g., $M \geqslant 2a_2\tau$). Then, $\forall x \geqslant M$ and $\forall t \leqslant \tau$

$$|j(t, x)| \leqslant |\phi(t, x)| \left|\frac{\partial}{\partial x}\phi(t, x)\right| \leqslant CC'(1 + |x| + |t|)^{-2}.$$

Therefore,

$$\lim_{M \to \infty} \left|\int_{-M}^{\tau} j(t, M)\,\mathrm{d}t\right| \leqslant \lim_{M \to \infty} \int_{-M}^{\tau} |j(t, M)|\,\mathrm{d}t$$

$$\leqslant CC' \lim_{M \to \infty} \int_{-M}^{\tau} (1 + |M| + |t|)^{-2}\,\mathrm{d}t$$

$$= CC' \lim_{M \to \infty} \left(\frac{1 + \mathrm{sgn}(\tau)}{1 + |M|} - \frac{1}{1 + 2|M|} - \frac{\mathrm{sgn}(\tau)}{1 + |M| + |\tau|}\right) = 0. \qquad \square$$

Now consider scattering solutions

$$\Psi(t, x) = \int_0^\infty \mathrm{d}k\varphi(k)\tilde{\phi}^{\mathrm{in}}(k; x)\, \mathrm{e}^{-\mathrm{i}k^2 t/2}$$

with $\tilde{\phi}^{\mathrm{in}}(k; x)$ a solution to the corresponding Lippman–Schwinger equation and φ again exclusively localized on the positive half-line. If the $\tilde{\phi}^{\mathrm{in}}(k; x)$ have the form

$$\tilde{\phi}^{\mathrm{in}}(k; x) = T(k)\, \mathrm{e}^{\mathrm{i}kx} \tag{A.4}$$

for $x > R$ for some $R > 0$, and $T \in C^1(\mathbb{R})$ (e.g., the potential barrier), then the above line of reasoning applies analogously. Clearly expression ($A.4$) is only valid for scattering potentials with support bounded from the right.

Appendix B. Proof of proposition 1.

By $\gamma_x : \mathbb{R} \to \mathbb{R}, t \mapsto \gamma_x(t)$, the integral curve of the Bohmian velocity vector field with initial datum x is denoted. The intervals $[a_t, b_t] := \{x \in \mathbb{R}/\gamma_x(t) \in [a, b]\}$ represent the initial data on configuration space, which are projected onto the interval $[a, b]$ at time t along their integral curves. Let $x_c \in \mathbb{R}$ be the initial condition of the trajectory, which separates the transmitted from the reflected ensemble. Then the transmission time (2) is

$$
\begin{aligned}
\langle \tau_T \rangle &= \int_{x_c}^{\infty} d\xi \, |\Psi(0, \xi)|^2 \int_{\tau_i}^{\tau_f} dt \, \chi_{[a,b]}(\gamma_\xi(t)) \\
&= \int_{\tau_i}^{\tau_f} dt \int_{x_c}^{\infty} d\xi \, |\Psi(0, \xi)|^2 \chi_{[a_t, b_t]}(\xi) \\
&= \int_{[\tau_i, \tau_f]} dt \int_{[a_t, b_t] \cap [x_c, \infty[} d\xi \, |\Psi(0, \xi)|^2 \\
&= \int_{[\tau_i, \tau_f]} dt \int_{[a, b]} d\xi \, |\Psi(t, \xi)|^2 \Theta(\xi - \gamma_{x_c}(t)).
\end{aligned}
$$

Analogously, the reflection time reads

$$
\langle \tau_R \rangle = \int_{[\tau_i, \tau_f]} dt \int_{[a, b]} d\xi \, |\Psi(t, \xi)|^2 \Theta(\gamma_{x_c}(t) - \xi).
$$

Now from the right-hand side of (5) together with (3) and (7) we get

$$
\begin{aligned}
\int_{\tau_i}^{\tau_f} dt &\left[\min \left\{ \int_a^{\infty} |\Psi(t, \xi)|^2 \, d\xi, \int_{\gamma_{x_c}(t)}^{\infty} |\Psi(t, \xi)|^2 \, d\xi \right\} \right. \\
&\left. - \min \left\{ \int_b^{\infty} |\Psi(t, \xi)|^2 \, d\xi, \int_{\gamma_{x_c}(t)}^{\infty} |\Psi(t, \xi)|^2 \, d\xi \right\} \right] \\
&= \int_{\tau_i}^{\tau_f} dt \left[\int_{\max\{a, \gamma_{x_c}(t)\}}^{\infty} |\Psi(t, \xi)|^2 \, d\xi - \int_{\max\{b, \gamma_{x_c}(t)\}}^{\infty} |\Psi(t, \xi)|^2 \, d\xi \right] \\
&= \int_{\tau_i}^{\tau_f} dt \int_{\max\{a, \gamma_{x_c}(t)\}}^{\max\{b, \gamma_{x_c}(t)\}} |\Psi(t, \xi)|^2 \, d\xi \\
&= \int_{\tau_i}^{\tau_f} dt \int_a^b |\Psi(t, \xi)|^2 \cdot \Theta(\xi - \gamma_{x_c}(t)) \, d\xi = \langle \tau_T \rangle.
\end{aligned}
$$

Analogously the right-hand side of (6) together with (3) and (7) becomes

$$
\begin{aligned}
\int_{\tau_i}^{\tau_f} dt &\left[\max \left\{ \int_a^{\infty} |\Psi(t, \xi)|^2 \, d\xi, \int_{\gamma_{x_c}(t)}^{\infty} |\Psi(t, \xi)|^2 \, d\xi \right\} \right. \\
&\left. - \max \left\{ \int_b^{\infty} |\Psi(t, \xi)|^2 \, d\xi, \int_{\gamma_{x_c}(t)}^{\infty} |\Psi(t, \xi)|^2 \, d\xi \right\} \right] \\
&= \int_{\tau_i}^{\tau_f} dt \int_{\min\{a, \gamma_{x_c}(t)\}}^{\min\{b, \gamma_{x_c}(t)\}} |\Psi(t, \xi)|^2 \, d\xi \\
&= \int_{\tau_i}^{\tau_f} dt \int_a^b |\Psi(t, \xi)|^2 \cdot \Theta(\gamma_{x_c}(t) - \xi) \, d\xi = \langle \tau_R \rangle
\end{aligned}
$$

which completes the proof.

References

[1] Hauge E H and Støvneng J A 1989 *Rev. Mod. Phys.* **61** 917
    Olkhovsky V S and Recami E 1992 *Phys. Rep.* **214** 339
    Landauer M and Martin Th 1994 *Rev. Mod. Phys.* **66** 217
    Olkhovsky V S and Recami E 1995 *J. Phys.* I **5** 1351
[2] Muga J G, Brouard S and Sala R 1992 *Phys. Lett.* A **167** 24
[3] Leavens C R 1990 *Solid State Commun.* **74** 923
[4] Damborenea J A, Egusquiza I L, Muga J G and Navarro B 2004 *Preprint* quant-ph/0403081
[5] Smith F T 1960 *Phys. Rev.* **118** 349
    Goldberger M L and Watson K M 1964 *Collision Theory* (New York: Wiley)
[6] Nussenzveig H M 1972 *Phys. Rev.* D **6** 1534
    Bollé D and Osborn T A 1976 *Phys. Rev.* D **13** 299
[7] Baz A I 1967 *Sov. J. Nucl. Phys.* **4** 182
[8] Martin Ph 1981 *Acta. Phys. Austriaca Suppl.* **23** 157
[9] Rybachenko V F 1967 *Sov. J. Nucl. Phys.* **5** 635
[10] Büttiker M and Landauer R 1982 *Phys. Rev. Lett.* **49** 1739
[11] Büttiker M 1983 *Phys. Rev.* B **27** 6178
[12] Brouard S, Sala R and Muga J G 1994 *Phys. Rev.* A **49** 4312
[13] Muga J G *et al* (ed) 2002 *Time in Quantum Mechanics* (Berlin: Springer)
[14] Dürr D, Goldstein S and Zanghi N 1992 *J. Stat. Phys.* **67** 843
    Berndl K, Daumer M, Dürr D, Goldstein S and Zanghi N 1995 *Nuovo Cimento* **110B** 737
[15] Stomphorst R G 2002 *Phys. Lett.* A **292** 213
[16] Kreidl S, Grübl G and Embacher H G 2003 *J. Phys. A: Math. Gen.* **36** 8851
[17] Reed M and Simon B 1979 *Methods of Modern Mathematical Physics* vol 3 (New York: Academic)
[18] Dollard J D 1969 *Commun. Math. Phys.* **12** 193
[19] Leavens C R 1990 *Solid State Commun.* **76** 253
[20] Oriols X, Martín F and Suñé J 1996 *Phys. Rev.* A **54** 2594
[21] Daumer M, Dürr D, Goldstein S and Zanghi N 1994 *On Three Levels: The Micro-, Meso-, and Macroscopic Approaches in Physics* ed M Fannes *et al* (New York: Plenum) p 331
[22] Mandez E E, Esaki L and Wang W I 1986 *Phys. Rev.* B **33** 2893
    Lee B 1993 *Superlatt. Microstruct.* **14** 295
    York J T, Coalson R D and Dahnovsky Y 2002 *Phys. Rev.* B **65** 235321